Stochastic bifurcation in FitzHugh-Nagumo ensembles subjected to additive and/or multiplicative noises

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Abstract

We have studied the dynamical properties of finite N-unit FitzHugh-Nagumo (FN) ensembles subjected to additive and/or multiplicative noises, reformulating the augmented moment method (AMM) with the Fokker-Planck equation (FPE) method [H. Hasegawa, J. Phys. Soc. Jpn. **75**, 033001 (2006)]. In the AMM, original $2N$-dimensional stochastic equations are transformed to eight-dimensional deterministic ones, and the dynamics is described in terms of averages and fluctuations of local and global variables. The stochastic bifurcation is discussed by a linear stability analysis of the *deterministic* AMM equations. The bifurcation transition diagram of multiplicative noise is rather different from that of additive noise: the former has the wider oscillating region than the latter. The synchronization in globally coupled FN ensembles is also investigated. Results of the AMM are in good agreement with those of direct simulations (DSs).

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1 INTRODUCTION

The FitzHugh-Nagumo (FN) model [1, 2] has been widely adopted as a simple model for a wide class of subjects not only for neural networks but also for reaction-diffusion chemical systems. Many studies have been made for the FN model with single elements [3]-[12] and globally-coupled ensembles [12]-[17]. The FN model is usually solved by direct simulation (DS) or the Fokker-Planck equation (FPE) method. For N-unit FN model, DS requires the computational time which grows as N^2 with increasing N. The FPE method leads to $(2N + 1)$-dimensional partial equations to be solved with appropriate boundary conditions. A useful semi-analytical method for stochastic equations has been proposed, taking account of the first and second moments of variables [5]. Recently we have proposed in [13, 16] the augmented moment method (AMM) based on a macroscopic point of view. In the AMM, we describe the properties of the stochastic ensembles in terms of a fairly small number of variables, paying our attention to their global behavior. For the N-unit stochastic systems, each of which is described by K variables, KN-dimensional stochastic equations are transformed to N_{eq}-dimensional *deterministic* equations in the AMM where $N_{eq} = K(K + 2)$ independent of N. This figure is, for example, $N_{eq} = 3$ for the Langevin model ($K = 1$) and $N_{eq} = 8$ for the FN model ($K = 2$). The AMM has been successfully applied to a study on the dynamics of coupled stochastic systems described by the Langevin, FN and Hodgkin-Huxley models subjected to additive noises with global, local or small-world couplings (with and without transmission delays) [13, 16, 18].

In recent years, the noise-induced phenomena such as stochastic resonance, noised-induced ordered state and noised-induced bifurcation have been extensively studied. Interesting phenomena caused by additive and multiplicative noises have been intensively investigated (for a recent review, see Refs. [19, 20]: related references therein). It has been realized that the properties of multiplicative noises are different from those of additive noises in some respects as follows. (1) Multiplicative noise induces the transition, creating an ordered state, while additive noise is against the ordering [21]-[29]. (2) Although the probability distribution in Langevin systems subjected to additive Gaussian noise follows the Gaussian, multiplicative Gaussian noise generally yields non-Gaussian distribution [30]-[33]. (3) The scaling relation of the effective strength: $\beta(N) = \beta(1)/\sqrt{N}$ valid for additive noise is not applicable for multiplicative noise: $\alpha(N) \neq \alpha(1)/\sqrt{N}$, where $\alpha(N)$ and $\beta(N)$ denote effective strengths of multiplicative and additive noises, respectively, of N-unit systems [34].

In order to show the above item (3), the present author has adopted the AMM for the Langevin model in a recent paper [34]. The AMM was originally developed by expanding variables around their mean values in stochastic equations to obtain the second-order moments both for local and global variables [13]. To extend the applicability of the AMM to stochastic systems including multiplicative noises, we have reformulated it for the Langevin model with the use of the FPE [34, 35]. It has been pointed out [34] that a naive approximation of the scaling relation for multiplicative noise: $\alpha(N) = \alpha(1)/\sqrt{N}$, as adopted in [29], leads to the result which is in disagreement with that of DSs.

It is doubly difficult to study analytically the *dynamical* properties of stochastic systems with *finite* populations. Most of analytical theories having been proposed so far are limited to infinite systems. Usually we solve the FPE for $N = \infty$ ensembles by using the mean-field and diffusion approximations to get the stationary probability distribution. For a study of dynamics, we have to obtain the instantaneous probability distribution from the partial differential equations (DEs) within the FPE, which is difficult even for $N = \infty$. Recently, the time-dependent probability distribution is treated with a series expansion of the Hermite polynomials, with which dynamics of $N = \infty$ stochastic systems is expressed by the time-dependent expansion coefficients [12]. In the AMM, equations of motions for N_{eq} moments which have clear physical meanings may describe the dynamics of stochastic systems with finite N.

In this paper, we will study effects of additive and/or multiplicative noises on the dynamical properties of the FN model. Although effects of additive noise on the FN model have been extensively investigated [3]-[17], there have been no such studies on the effect of multiplicative noise, as far as the author is concerned. We are interested in the stochastic bifurcation, which is one of interesting phenomena induced by noise (Refs. [36, 37], related references therein). The theory on stochastic bifurcation is still in its infancy. Indeed, there is no stringent definition of the stochastic bifurcation. At the moment, two kinds of definitions have been proposed: (i) one is based on a sudden change in the stationary probability distribution, and (ii) the other is based on a sudden change in the sign of the largest Lyapunov index. Unfortunately these two definitions do not necessarily yield the same result. The bifurcation of the single [11] and ensemble FN model [12, 17] subjected to additive noise has been recently discussed. Based on the second-order moment method, the bifurcation analysis has been made for globally-coupled FN model in Ref. [17], where dynamics of fast variables is separated from and projected to that of slow variables subjected to additive noise. We will discuss the bifurcation in the

FN ensembles subjected to additive and/or multiplicative noises, making a linear stability analysis to our *deterministic* AMM equations. It is much easier to study the deterministic DEs than stochastic DEs.

The purpose of the present paper is two folds: (i) to reformulate AMM for the FN model subjected to additive and multiplicative noises with the use of FPE [34, 35], and (ii) to discuss the respective roles of additive and multiplicative noises on the stochastic bifurcation and synchronization. The paper is organized as follows. In Sec. II, we will apply the AMM to finite N-unit FN ensembles subjected to additive and multiplicative noises. With the use of the AMM equations, the stochastic bifurcation is discussed in Sec. III. Some discussions on the synchronization are presented in Sec. IV. The final Sec. V is devoted to conclusion.

2   FN neuron ensembles

2.1   Description of our model

We have adopted N-unit stochastic systems described by the FN model subjected to additive and multiplicative noises. Dynamics of the coupled ensemble is expressed by nonlinear DEs given by

$$\frac{dx_i}{dt} = F(x_i) - c\,y_i + \alpha\,G(x_i)\eta_i(t) + \beta\,\xi_i(t) + I_i^{(c)}(t) + I^{(e)}(t), \tag{1}$$

$$\frac{dy_i}{dt} = b\,x_i - d\,y_i + e, \qquad (i = 1 \text{ to } N) \tag{2}$$

with

$$I_i^{(c)}(t) = \frac{J}{Z}\sum_{j \neq i}(x_j - x_i). \tag{3}$$

In Eq. (1)-(3), $F(x) = a_3 x^3 + a_2 x^2 + a_1 x$, $a_3 = -0.5$, $a_2 = 0.55$, $a_1 = -0.05$, $b = 0.015$, $c = 1.0$, $d = 0.003$ and $e = 0$ [5, 13]: x_i and y_i denote the fast and slow variables, respectively: $G(x)$ is an arbitrary function of x: $I^{(e)}(t)$ stands for an external input: J expresses the strength of diffusive couplings, $Z = N - 1$: α and β denote magnitudes of multiplicative and additive noises, respectively, and $\eta_i(t)$ and $\xi_i(t)$ express zero-mean Gaussian white noises with correlations given by

$$\langle \eta_i(t)\,\eta_j(t') \rangle = \delta_{ij}\delta(t - t'), \tag{4}$$

$$\langle \xi_i(t)\,\xi_j(t') \rangle = \delta_{ij}\delta(t - t'), \tag{5}$$

$$\langle \eta_i(t)\,\xi_j(t') \rangle = 0. \tag{6}$$

The Fokker-Planck equation $p(\{x_i\}, \{y_i\}, t)$ is expressed in the Stratonovich representation by[34][38]

$$\frac{\partial}{\partial t}p = -\sum_k \frac{\partial}{\partial x_k}\{[F(x_k) - cy_k + I_k]p\} - \sum_k \frac{\partial}{\partial y_k}[(bx_k - dy_k + e)p]$$
$$+ \left(\frac{\alpha^2}{2}\right)\sum_k \frac{\partial}{\partial x_k}\{G(x_k)\frac{\partial}{\partial x_k}[G(x_k)\,p]\} + \sum_k \left(\frac{\beta^2}{2}\right)\frac{\partial^2}{\partial x_k^2}\,p, \quad (7)$$

where $I_k = I_k^{(c)} + I^{(e)}$.

We are interested also in dynamics of global variables $X(t)$ and $Y(t)$ defined by

$$X(t) = \frac{1}{N}\sum_i x_i(t), \quad (8)$$

$$Y(t) = \frac{1}{N}\sum_i y_i(t). \quad (9)$$

The probability of $P(X, Y, t)$ is expressed in terms of $p(\{x_i\}, \{y_i\}, t)$ by

$$P(X, Y, t) = \int\int \Pi_i dx_i dy_i\; p_i(\{x_i\}, \{y_i\}, t)\, \delta\left(X - \frac{1}{N}\sum_i x_i\right)\delta\left(Y - \frac{1}{N}\sum_i y_i\right). \quad (10)$$

Moments of local and global variables are expressed by

$$\langle x_i^k\, y_i^\ell\rangle = = \int\int dx_i dy_y\; p_i(x_i, y_i, t)\, x_i^k y_i^\ell, \quad (11)$$

$$\langle X^k\, Y^\ell\rangle = \int\int dX dY\, P(X, Y, t)\, X^k Y^\ell. \quad (12)$$

By using Eqs. (1), (2), (7) and (11), we get equations of motions for means, variances and covariances of local variables by

$$\frac{d\langle x_i\rangle}{dt} = \langle F(x_i)\rangle - c\langle y_i\rangle + \frac{\alpha^2}{2}\langle G'(x_i)G(x_i)\rangle, \quad (13)$$

$$\frac{d\langle y_i\rangle}{dt} = b\langle x_i\rangle - d\langle y_i\rangle + e, \quad (14)$$

$$\frac{d\langle x_i x_j\rangle}{dt} = \langle x_i F(x_j)\rangle + \langle x_j F(x_i)\rangle - c(\langle x_i y_j\rangle + \langle x_j y_i\rangle)$$
$$+\frac{J}{Z}\sum_k(\langle x_i x_k\rangle + \langle x_j x_k\rangle - \langle x_i^2\rangle - \langle x_j^2\rangle)$$
$$+\frac{\alpha^2}{2}[\langle x_i G'(x_j)G(x_j)\rangle + \langle x_j G'(x_i)G(x_i)\rangle]$$
$$+[\alpha^2\langle G(x_i)^2\rangle + \beta^2]\delta_{ij}, \quad (15)$$

$$\frac{d\langle y_i y_j\rangle}{dt} = b(\langle x_i y_j\rangle + \langle x_j y_i\rangle) - 2d\langle y_i y_j\rangle, \quad (16)$$

$$\frac{d\langle x_i\, y_j\rangle}{dt} = \langle y_j\, F(x_i)\rangle - c\langle y_i y_j\rangle + b\langle x_i x_j\rangle - d\langle x_i\, y_j\rangle$$
$$+\frac{w}{Z}\sum_k(\langle x_k\, y_j\rangle - \langle x_i\, y_j\rangle) + \frac{\alpha^2}{2}\langle y_j\, G'(x_i)G(x_i)\rangle, \quad (17)$$

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where $G'(x) = dG(x)/dx$. Equations (13)-(17) may be obtainable also with the use of the Furutsu-Novikov theorem [39, 40].

From Eqs. (8), (9) and (12), we get equations of motions for variances and covariances of global variables:

$$\frac{d\langle V_\kappa \rangle}{dt} = \frac{1}{N} \sum_i \langle v_{\kappa i} \rangle, \tag{18}$$

$$\frac{d\langle V_\kappa V_\lambda \rangle}{dt} = \frac{1}{N^2} \sum_i \sum_j \frac{d\langle v_{\kappa i} v_{\lambda j} \rangle}{dt}, \qquad (\kappa, \gamma = 1, 2) \tag{19}$$

where we adopt a convention: $v_{1i} = x_i$, $v_{2i} = y_i$, $V_1 = X$ and $V_2 = Y$. Equations (13) and (14) were used for single ($N = 1$) and infinite ($N = \infty$) FN neurons subjected only to additive noise ($\alpha = 0$) in the mean-field approximation [12]. Equations (13)-(17) were employed in the moment method for a single FN neuron subjected to additive noises [5]. We will show that Eqs. (18) and (19) play important roles in discussing finite N-unit FN ensembles.

2.2 AMM equations

In the AMM [13], we define eight quantities given by

$$\mu_\kappa = \langle V_\kappa \rangle = \frac{1}{N} \sum_i \langle v_{\kappa i} \rangle, \tag{20}$$

$$\gamma_{\kappa, \lambda} = \frac{1}{N} \sum_i \langle (v_{\kappa i} - \mu_\kappa)(v_{\lambda i} - \mu_\lambda) \rangle, \tag{21}$$

$$\rho_{\kappa, \lambda} = \langle (V_\kappa - \mu_\kappa)(V_\lambda - \mu_\lambda) \rangle, \qquad (\kappa, \lambda = 1, 2) \tag{22}$$

with $\gamma_{1,2} = \gamma_{2,1}$ and $\rho_{1,2} = \rho_{2,1}$. It is noted that $\gamma_{\kappa, \lambda}$ expresses the averaged fluctuations of local variables while $\rho_{\kappa, \lambda}$ denotes those of global variables.

Expanding Eqs. (13)-(19) around means of μ_κ as $v_{\kappa i} = \mu_\kappa + \delta v_{\kappa i}$ and retaining the terms of $O(\langle \delta v_{\kappa i} \delta v_{\lambda j} \rangle)$, we get equations of motions for the eight quantities given by

$$\frac{d\mu_1}{dt} = f_o + f_2 \gamma_{1,1} - c\mu_2 + \frac{\alpha^2}{2}[g_0 g_1 + 3(g_1 g_2 + g_0 g_2)\gamma_{1,1}] + I^{(e)}, \tag{23}$$

$$\frac{d\mu_2}{dt} = b\mu_1 - d\mu_2 + e, \tag{24}$$

$$\frac{d\gamma_{1,1}}{dt} = 2(a\gamma_{1,1} - c\gamma_{1,2}) + \frac{2JN}{Z}(\rho_{1,1} - \gamma_{1,1}) + 2\alpha^2(g_1^2 + 2g_0 g_3)\gamma_{1,1}$$
$$+ \alpha^2 g_0^2 + \beta^2, \tag{25}$$

$$\frac{d\gamma_{2,2}}{dt} = 2(b\gamma_{1,2} - d\gamma_{2,2}), \tag{26}$$

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$$\frac{d\gamma_{1,2}}{dt} = b\gamma_{1,1} + (a-d)\gamma_{1,2} - c\gamma_{2,2} + \frac{JN}{Z}(\rho_{1,2} - \gamma_{1,2}) + \frac{\alpha^2}{2}(g_1^2 + 2g_0g_2)\gamma_{1,2}, \quad (27)$$

$$\frac{d\rho_{1,1}}{dt} = 2(a\rho_{1,1} - c\rho_{1,2}) + 2\alpha^2(g_1^2 + 2g_0g_2)\rho_{1,1} + \frac{\alpha^2 g_0^2}{N} + \frac{\beta^2}{N}, \quad (28)$$

$$\frac{d\rho_{2,2}}{dt} = 2(b\rho_{1,2} - d\rho_{2,2}), \quad (29)$$

$$\frac{d\rho_{1,2}}{dt} = b\rho_{1,1} + (a-d)\rho_{1,2} - c\rho_{2,2} + \frac{\alpha^2}{2}(g_1^2 + 2g_0g_2)\rho_{1,2}, \quad (30)$$

with

$$a = f_1 + 3f_3\gamma_{1,1}, \quad (31)$$

$$f_\ell = (1/\ell\,!)F^{(\ell)}(\mu_1), \quad (32)$$

$$g_\ell = (1/\ell\,!)G^{(\ell)}(\mu_1). \quad (33)$$

The original $2N$-dimensional stochastic equations given by Eqs. (1)-(3) are transformed to eight-dimensional deterministic equations. Equations (23)-(30) with additive noises only ($\alpha = 0$) reduce to those obtained previously [13]. We note that in the limit of $J = 0$, AMM equations lead to

$$\rho_{\kappa,\lambda} = \frac{\gamma_{\kappa,\lambda}}{N}, \qquad (\kappa, \lambda = 1, 2) \quad (34)$$

which is nothing but the central-limit theorem describing the relation between fluctuations in local and average variables. In the limit of $N = 1$, we get $\rho_{\kappa,\lambda} = \gamma_{\kappa,\lambda}$, by which the AMM equations reduces to the five-dimensional DEs for μ_1, μ_2, $\gamma_{1,1}$, $\gamma_{2,2}$ and $\gamma_{1,2}$.

Equations (1)-(3) for $I^{(e)} = \alpha = \beta = 0$ have the stationary solution of $x_i = y_i = 0$. When we assume $G(x)$ for the multiplicative noise given by

$$G(x) = x, \quad (35)$$

the AMM equations are expressed by

$$\frac{d\mu_1}{dt} = f_o + f_2\gamma_{1,1} - c\mu_2 + \frac{\alpha^2\mu_1}{2} + I^{(e)}, \quad (36)$$

$$\frac{d\mu_2}{dt} = b\mu_1 - d\mu_2 + e, \quad (37)$$

$$\frac{d\gamma_{1,1}}{dt} = 2(a\gamma_{1,1} - c\gamma_{1,2}) + \frac{2JN}{Z}(\rho_{1,1} - \gamma_{1,1}) + 2\alpha^2\gamma_{1,1} + \alpha^2\mu_1^2 + \beta^2, \quad (38)$$

$$\frac{d\gamma_{2,2}}{dt} = 2(b\gamma_{1,2} - d\gamma_{2,2}), \quad (39)$$

$$\frac{d\gamma_{1,2}}{dt} = b\gamma_{1,1} + (a-d)\gamma_{1,2} - c\gamma_{2,2} + \frac{JN}{Z}(\rho_{1,2} - \gamma_{1,2}) + \frac{\alpha^2\gamma_{1,2}}{2}, \quad (40)$$

$$\frac{d\rho_{1,1}}{dt} = 2(a\rho_{1,1} - c\rho_{1,2}) + 2\alpha^2\rho_{1,1} + \frac{\alpha^2\mu_1^2}{N} + \frac{\beta^2}{N}, \tag{41}$$

$$\frac{d\rho_{2,2}}{dt} = 2(b\rho_{1,2} - d\rho_{2,2}), \tag{42}$$

$$\frac{d\rho_{1,2}}{dt} = b\rho_{1,1} + (a - d)\rho_{1,2} - c\rho_{2,2} + \frac{\alpha^2\rho_{1,2}}{2}. \tag{43}$$

The stochastic bifurcation of the AMM equations given by Eqs. (36)-(43) will be investigated in Sec. 3. AMM equations for a more general form of $G(x) = x \mid x \mid^{s-1}$ ($s \geq 0$) are presented in Appendix A. Some numerical results for various s values will be discussed in Sec. 4 (Fig. 13).

2.3 Properties of the AMM

Contributions from multiplicative noise have the more complicated N dependence than those from additive noise. Comparing the β^2 term in $d\gamma_{1,1}/dt$ of Eq. (25) to that in $d\rho_{1,1}/dt$ of Eq. (28), we note that the effective strength of additive noise of the N-unit system, $\beta(N)$, is scaled by

$$\beta(N) = \frac{\beta(1)}{\sqrt{N}}. \tag{44}$$

In contrast, a comparison between the α^2 terms in Eq. (25) and (28) yield the two kinds of scalings:

$$\alpha(N) = \frac{\alpha(1)}{\sqrt{N}}, \qquad \text{for } \mu_1 \text{ term}, \tag{45}$$

$$\alpha(N) = \alpha(1), \qquad \text{for } \gamma_{1,1} \text{ and } \rho_{1,1} \text{ terms}, \tag{46}$$

The relations given by Eqs. (45) and (46) hold also for $d\gamma_{1,2}/dt$ and $d\rho_{1,2}/dt$ given by Eqs. (27) and (30). Thus the scaling behavior of the effective strength of multiplicative noise is quite different from that of additive noise, as previously pointed out for Langevin model [34]. If the relations: $\alpha(N) = \alpha(1)/\sqrt{N}$ and $\beta(N) = \beta(1)/\sqrt{N}$ hold, the FPE for $P(X, Y, t)$ of the global variables of X and Y in N-unit systems may be expressed by

$$\frac{\partial}{\partial t}P(X, Y, t) = -\frac{\partial}{\partial X}\{[F(X) - cY + I]P(X, Y, t)\} - \frac{\partial}{\partial Y}[(bX - dY + e)P(X, Y, t)]$$
$$+ \frac{\alpha^2}{2N}\frac{\partial}{\partial X}\{G(x)\frac{\partial}{\partial X}[G(X)\,P(X, Y, t)]\} + \frac{\beta^2}{2N}\frac{\partial^2}{\partial X^2}\,P(X, Y, t). \tag{47}$$

Unfortunately it is not the case as shown in Eq. (46), although Eq. (47) may be valid in the case of additive noise only ($\alpha = 0$) [12].

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3 STOCHASTIC BIFURCATION

3.1 Additive versus multiplicative noises

3.1.1 The case of $N = 1$

In order to get an insight to the AMM, we first examine the case of single element ($N = 1$), for which AMM equations for μ_1, μ_2, $\gamma_{1,1}$, $\gamma_{2,2}$ and $\gamma_{1,2}$ are given by

$$\frac{d\mu_1}{dt} = f_o + f_2\gamma_{1,1} - c\mu_2 + \frac{\alpha^2\mu_1}{2} + I^{(e)}, \tag{48}$$

$$\frac{d\mu_2}{dt} = b\mu_1 - d\mu_2 + e, \tag{49}$$

$$\frac{d\gamma_{1,1}}{dt} = 2(a\gamma_{1,1} - c\gamma_{1,2}) + 2\alpha^2\gamma_{1,1} + \alpha^2\mu_1^2 + \beta^2, \tag{50}$$

$$\frac{d\gamma_{2,2}}{dt} = 2(b\gamma_{1,2} - d\gamma_{2,2}), \tag{51}$$

$$\frac{d\gamma_{1,2}}{dt} = b\gamma_{1,1} + (a - d)\gamma_{1,2} - c\gamma_{2,2} + \frac{\alpha^2\gamma_{1,2}}{2}, \tag{52}$$

where $a = f_1 + 3f_3\gamma_{1,1}$. We have applied a step input given by

$$I^{(e)}(t) = A\,\Theta(t - t_{in}), \tag{53}$$

where $A = 0.1$, $t_{in} = 50$ and $\Theta(x)$ denotes the Heaviside function: $\Theta(x) = 1$ for $x \geq 0$ and zero otherwise. Equations (48)-(52) for $N = 1$ have been solved by using the fourth-order Runge-Kutta method with a time step of 0.01 and with zero initial data. Direct simulations (DSs) for the N-unit FN model given by Eqs. (1)-(3) have been performed by using the Heun method with a time step of 0.003 and with the initial data of $x_i(0)$ and $y_i(0)$ which are randomly chosen from $[-0.01, 0.01]$. Results of DS are averaged over 1000 trials otherwise noticed. All quantities are dimensionless.

Figure 1 shows time courses of $\mu_1(t)$, $\mu_2(t)$, $\gamma_{1,1}(t)$, $\gamma_{2,2}(t)$ and $\gamma_{1,2}(t)$ calculated by AMM (solid curves) and DS (dashed curves) with $\alpha = 0.01$, $\beta = 0.0$ ($N = 1$). A single FN neuron fires when the external input $I^{(e)}(t)$ is applied for $t \geq 50$. By an applied input, μ_1, μ_2, $\gamma_{1,1}$, $\gamma_{2,2}$ and $\gamma_{1,2}$ show the time-dependent behavior, and they approach the stationary values at $t > 300$. Results calculated by AMM are in good agreement with those of DS.

Time courses of $\mu_1(t)$ [$\gamma_{1,1}(t)$] with $\alpha = 0.05$, 0.1, 0.2 and 0.5 are plotted in Fig. 2(a)-(d) [Fig. 2(e)-2(h)] when an applied input given by Eq. (53) is applied. With increasing α, the magnitude of $\gamma_{1,1}(t)$ is much increased, although the profile of $\mu_1(t)$ is almost the same except for $\alpha = 0.5$. The time course of $\mu_1(t)$ for $\alpha = 0.5$ in the AMM shows an

oscillation which is expected to be due to the stochastic bifurcation. Although the result of DS averaged over 1000 trials shows no oscillation, that of a single trial clearly shows the oscillation.

As was shown in Fig. 2(d) and 2(h), the bifurcation may be induced by strong noise. In order to discuss the bifurcation, we have applied a constant input given by

$$I^{(e)}(t) = I. \tag{54}$$

The stationary equations given by Eqs. (48)-(52) with $d\mu_1/dt = 0$ *et al.* are solved by the Newton-Raphson method. Then dynamics given by Eqs. (48)-(52) is solved with initial values of the stationary solutions. Figures 3(a) and 3(b) show the time courses of $\mu_1(t)$ for inputs of $I = 0.1$ and $I = 0.5$, respectively, with multiplicative noise ($\alpha = 0.1$ and $\beta = 0.0$): solid and dashed curve express the results of the AMM and DS with a single trail, respectively. We note that for $I = 0.5$, μ_1 begins to oscillate and its magnitude is gradually increased while for $I = 0.1$, μ_1 shows no time development. This is more clearly seen in the μ_1-μ_2 plots shown in Fig. 3(b) and 3(d). The oscillation for $I = 0.5$ is due to the stochastic bifurcation. Figures 3(e) and 3(f) will be explained later in Sec. 3.1.2.

We have calculated the bifurcation transition diagrams by making a linear stability analysis to the *deterministic* AMM equations. The 5×5 Jacobian matrix C of the AMM equations (39)-(43) is expressed with a basis of (μ_1, μ_2, $\gamma_{1,1}$, $\gamma_{2,2}$, $\gamma_{1,2}$) by

$$C = \begin{bmatrix} f_0' + f_2'\gamma_{1,1} + \alpha^2/2 & -c & f_2 & 0 & 0 \\ b & -d & 0 & 0 & 0 \\ 2[(f_1' + 3f_3'\gamma_{1,1})\gamma_{1,1} + \alpha^2\mu_1] & 0 & 2(f_1 + 6f_3\gamma_{1,1} + \alpha^2) & 0 & -2c \\ 0 & 0 & 0 & -2d & 2b \\ (f_1' + 3f_3'\gamma_{1,1})\gamma_{1,2} & 0 & b + 3f_3\gamma_{1,2} & -c & a - d + \alpha^2/2 \end{bmatrix}, \tag{55}$$

where $a = f_1 + 3f_3\gamma_{1,1}$ and $f_k' = df_k/d\mu_1$. The instability is realized when any of real parts of five eigenvalues in the Jacobian matrix C is positive. For deterministic FN neuron without noises ($\alpha = \beta = 0.0$), the critical condition is given by $f_0' - d = 3\,a_3\mu_1^2 + 2\,a_2\mu_1 + a_1 - d = 0$ for the stationary μ_1, from which the oscillating state is realized for $0.26 < I < 3.34$.

Multiplicative noise

In order to obtain the transition diagram, we have performed calculations of the stationary state and eigenvalues of its Jacobian matrix C, by sequentially changing a model parameter such as α, β, J and I. A calculation of the stationary state for a given α value, for example, has been made by the Newton-Raphon method with initial values

which are given from a calculation for the preceding α value. Figure 4(a) shows the I-α transition diagram obtained for multiplicative noise ($\beta = 0.0$, $N = 1$). When changing a parameter as mentioned above, we have the continuous (second-order) and discontinuous (first-order) transitions. Solid curves denote the boundaries of the second-order transition between the oscillating (OSC) and non-oscillating (NONOSC). As for the first-order transition, at $I = 2.0$, for example, the OSC$\rightarrow$NONOSC transition takes place at $\alpha = 0.11$ when α is increased from below, while the NONOSC$\rightarrow$OSC transition occurs at $\alpha = 0.04$ when α is decreased from above. The state for $0.04 < \alpha < 0.11$ with hysteresis is referred to as the OSC$'$ state hereafter. In the OSC$'$ state, we have two stationary solutions, as will be discussed below.

Dashed curves in Fig. 5(a) show the I dependence of the maximum real part among five eigenvalues, λ_m (referred to as the *maximum index*), for multiplicative noise ($\alpha = 0.1$, $\beta = 0.0$). Two dashed curves for $0.19 < I < 2.29$ express the results of the two stationary solutions in the OSC$'$ state shown in Fig. 4(a). The lower, dashed curve crosses the zero line at four points at $I = 0.29$, 1.41, 2.39 and 3.41. For a comparison, the result for the deterministic model ($\alpha = \beta = 0.0$) is plotted by the chain curve. Although the OSC state disappears for fairly strong multiplicative noises of $\alpha > 0.16$ at $2 \stackrel{<}{\sim} I \stackrel{<}{\sim} 3.3$, it persists to strong noises at $0 \stackrel{<}{\sim} I \stackrel{<}{\sim} 1.3$.

Solid and dashed curves in Fig. 6(a) show μ_1 and $\gamma_{1,1}$, respectively, of the stationary state as a function of I. Two dashed curves for $0.19 < I < 2.29$ express $\gamma_{1,1}$ of the two stationary solutions in the OSC$'$ state. In contrast, there is little difference in μ_1 for the two stationary solutions. We note that μ_1 is nearly proportional to I and that the upper curve of $\gamma_{1,1}$ has a broad peak centered at $I \sim 2$. Open and filled circles in Fig. 4(a) denote two sets of parameters of $(I, \alpha) = (0.1, 0.1)$ and $(0.5, 0.1)$ adopted for the calculations shown in Figs. 3(a) and (b).

Additive noise

Figure 4(b) shows the I-β transition diagram for additive noise ($\alpha = 0.0$, $N = 1$). Solid curve expresses the boundary between the OSC and NONOSC states, and dashed curve denotes the boundary of the first-order transition relevant to the OSC$'$ state with hysteresis. When an additive noise is included, the boundary for the OSC-NONOSC states is much modified. This is explained for the case of $\beta = 0.1$ in Fig. 5(b), where the chain curve denotes the maximum index λ_m for the case of $\alpha = \beta = 0.0$ while the dashed curve shows the result for additive noise of $\beta = 0.1$ ($\alpha = 0.0$). The chain curve crosses

the zero line at $I = 0.26$ and 3.34, while the dashed curve has the zeros at four points at $I = 0.12$, 0.86, 2.75 and 3.48. The transition diagram has a strange shape, which is symmetric with respect to the axis of $I = 1.80$. The I dependences of μ_1 and $\gamma_{1,1}$ are plotted by solid and dashed curves, respectively, in Fig. 6(b). $\gamma_{1,1}$ for additive noise has a broad peak similar to that for multiplicative noise shown in Fig. 6(a). It is noted that $\sqrt{\gamma_{1,1}}$ expresses the effective width of the probability distribution of $p(x) = \int p(x, y)\, dy$. We note that $\gamma_{1,1}$ in Fig. 6(a) shows a rapid increase at $I \sim 0.2$ where the OSC-NONOSC transition takes place in Fig. 4(a). Except this case, however, there are no abrupt changes in $\gamma_{1,1}$ at the transition.

3.1.2   The case of finite N

Multiplicative noise

Figure 3(e) shows the time courses of $\mu_1(t)$ for an input of $I = 0.5$ given by Eq. (54) applied to $N = 100$ FN ensembles with $J = 1.0$ subjected to multiplicative noise ($\alpha = 0.1$, $\beta = 0.0$): solid and dashed curves express the results of the AMM and DS with a single trail, respectively. It shows that μ_1 begins to oscillate and its magnitude is rapidly increased. This is more clearly seen in the μ_1-μ_2 plots shown in Fig. 3(f). In contrast, for smaller $I = 0.1$, we have not obtained the oscillating solution, just as in the $N = 1$ case shown in Figs. 3(a) and 3(b).

We have performed a linear analysis for the case of finite N, by using the 8×8 Jacobian matrix with the stationary solutions obtained from Eqs. (36)-(43), expressions for 8×8 Jacobian matrix elements being presented in Appendix B.

The I-α transition diagram for multiplicative noise with $N = 100$ and $J = 1.0$ is shown in Fig. 4(c). The region of the OSC state is a little decreased for a weak α but it is widen for stronger α. The solid curve in Fig. 5(a) expresses the I dependence of the maximum index λ_m for multiplicative noise with $\alpha = 0.1$ and $J = 1.0$, which crosses the zero line at two points at $I = 0.21$ and 3.37.

Figure 7(a) shows the J-α transition diagram for multiplicative noise ($\beta = 0.0$) for $I = 3.0$ with $N = 100$. For $\alpha = 0.1$, the OSC state is realized even for $J = 0.0$. For strong multiplicative noise of $\alpha = 0.3$, the OSC state disappears at $J \leq 0.365$. For $\alpha = 0.2$, the OSC state is realized not only at $J \geq 0.194$ but also at $0.085 \leq J \leq 0.136$, as shown in the inset of Fig. 7(a). This implies the re-entrance to the OSC state from the NONOSC state when the coupling is decreased from above.

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Additive noise

Figure 4(d) shows the I-β transition diagram for additive noise for $J = 1.0$ and $N = 100$. The width of the OSC state is gradually decreased with increasing β. Chain and solid curves in Fig. 5(b) show the I dependences of λ_m for $\beta = 0.0$ and $\beta = 0.1$, respectively, with $J = 1.0$ and $N = 100$. The former has the zeros at $I = 0.26$ and 3.34 while the latter at $I = 0.29$ and 3.32.

Figure 7(b) shows the J-β transition diagram for additive noise for $I = 3.0$ with $N = 100$. The critical values of β are 0.114, 0.221 and 0.265 for $J = 0.0$, 0.5 and 1.0, respectively. The region of the OSC state is increased with increasing J.

When we compare the bifurcation transition diagrams for multiplicative noise in Fig. 4(a) and 4(c) with those for additive noise in Fig. 4(b) and 4(d), we note that the former is rather different from the latter, in particular for the $N = 1$ case. When a weak additive noise is added to the $N = 1$ model, the OSC state is slightly increased although for a large noise, the OSC state disappears. In contrast, the OSC state persists for multiplicative noise. Figures 7(a) and 7(b) show that the coupling is beneficial to the OSC state for both additive and multiplicative noises, as expected.

3.2   Coexistence of additive and multiplicative noises

Although we have so far discussed the additive and multiplicative noises separately, we now consider the case where both the noises coexist. Figure 8(a) shows the $I - \alpha$ transition diagram for $\beta = 0.05$ and $N = 1$. It is similar to the transition diagram shown in Fig. 4(a) for the case of multiplicative noise only ($\beta = 0.0$). In Fig. 8(a), the OSC state is completely split with a gap even for $\alpha = 0.0$. This is because the OSC state disappears for additive noise of $\beta = 0.05$ even without multiplicative noise as shown in Fig. 4(b).

In contrast, Fig. 8(b) shows the $I - \beta$ transition diagram for $\alpha = 0.1$ and $N = 1$. We have the OSC$'$ state for weak β at $1.30 < I < 2.38$. This is related to the fact that the OSC$'$ state exists for $\alpha = 0.1$ and $\beta = 0.0$ in Fig. 4(a).

Figure 9 shows the α-β transition diagram for the OSC and NONOSC states with $I = 1.0$ and $I = 3.0$ ($N = 1$). In the case of $I = 1.0$, the boundary between the OSC and NONOSC states extends to large α, reflecting the behavior shown in Figs. 4(a) and 8(a). We have tried to fit the calculated boundaries by a simple expression of $\beta = c\sqrt{1 - (\alpha/d)^2}$. Dashed curves in Fig. 9 express the results with $c = 0.084$ and $d = 0.32$ for $I = 1.0$ and with $c = 0.121$ and $d = 0.158$ for $I = 3.0$. An agreement between the solid and dashed curves in the case of $I = 3.0$ is better than that in the case of $I = 1.0$.

4 DISCUSSION

It is interesting to study the synchronization in FN ensembles with noises. In order to quantitatively discuss the synchronization in the ensemble, we first consider the quantity given by [13]

$$R(t) = \frac{1}{N^2} \sum_{ij} \langle [x_i(t) - x_j(t)]^2 \rangle = 2[\gamma_{1,1}(t) - \rho_{1,1}(t)]. \tag{56}$$

When all neurons are in the completely synchronous state, we get $x_i(t) = X(t)$ for all i, and then $R(t) = 0$ in Eq. (56). On the contrary, in the asynchronous state, we get $R(t) = 2(1 - 1/N)\gamma_{1,1} \equiv R_0(t)$ because $\rho_{1,1} = \gamma_{1,1}/N$ [Eq. (34)]. We modify $R(t)$ such that the synchronization is scaled between the zero and unity, defining the synchronization ratio given by [13]

$$S(t) = 1 - \frac{R(t)}{R_0(t)} = \left[\frac{N\rho_{1,1}(t)/\gamma_{1,1}(t) - 1}{N - 1} \right], \tag{57}$$

which is 0 and 1 for completely asynchronous $(R = R_0)$ and synchronous states $(R = 0)$, respectively. As will be shown below, $S(t)$ depends not only on model parameters such as J and N but also the type of noises (α and β).

The calculations have been performed for an external input $I^{(e)}(t)$ given by Eq. (53). DS calculations have been made with 20 trials.

Figures 10(a)-(d) show time courses of $\mu_1(t)$, $\gamma_{1,1}(t)$, $\rho_{1,1}(t)$ and $S(t)$ for multiplicative noise ($\alpha = 0.01$, $\beta = 0.0$) with $J = 1.0$ and $N = 100$. In contrast, Figs. 10(e)-(h) express the result for additive noise ($\alpha = 0.01$, $\beta = 0.0$). When the external input is applied at $t = 50$, FN neurons fire, and $\gamma_{1,1}(t)$, $\rho_{1,1}(t)$ and $S(t)$ develop. The time dependence of $\mu_1(t)$ for multiplicative noise in Fig. 10(a) is almost the same as that for additive noises shown in Fig. 10(e). From a comparison between Figs. 10(b) and 10(f), we note, however, that $\gamma_{1,1}$ for multiplicative noise is considerably different from that for additive noise. This is true also for $\rho_{1,1}$ shown in Figs. 10(c) and 9(g). The time course of $S(t)$ is calculated with the use of Eq. (57) with $\gamma_{1,1}$ and $\rho_{1,1}$ shown in Figs. 10(b) and 10(c) [Figs. 10(f) and 10(g)]. Reflecting the differences in $\gamma_{1,1}$ and $\rho_{1,1}$, the result of $S(t)$ for multiplicative noise in Fig. 10(d) is quite different from that for additive noise in Fig. 10(h), although at $t > 400$, both $S(t)$ approach the same value of $S = 0.24$.

We may apply also spike train and sinusoidal inputs to the systems, whose results will be discussed in the followings. Figures 11(a) and 11(b) show time courses of $\mu_1(t)$ and $S(t)$ in the case of multiplicative noise ($\alpha = 0.01$, $\beta = 0.0$) with $J = 1.0$ and $N = 100$,

when a spike train input given by

$$I^{(e)}(t) = A \sum_n \Theta(t - t_n)\Theta(t_n + T_w - t), \tag{58}$$

with $A = 0.1$, $T_n = 50 + 100(n - 1)$ and $T_w = 10$ is applied, an input being plotted at the bottom of Fig. 11(a). In order to understand the relation between μ_1 and S, we depict the $S - (d\mu_1/dt)$ plot shown by solid curves in Fig. 11(c), where initial data of S and $d\mu_1/dt$ at $t < 350$ are neglected. We note that S is large for $d\mu_1/dt \sim -0.15$ and $0 < d\mu_1/dt < 0.1$. In contrast, we show by the dashed curve in Fig. 11(c), a similar plot in the case of additive noise ($\alpha = 0.0$, $\beta = 0.01$). The trend of multiplicative noise is similar to that of additive noise for $d\mu_1/dt < 0$ but quite different for $d\mu_1/dt > 0$.

Figures 12(a) and 12(b) show time courses of $\mu_1(t)$ and $S(t)$ in the case of multiplicative noise ($\alpha = 0.01$, $\beta = 0.0$ with $J = 1.0$ and $N = 100$, when a sinusoidal input given by

$$I^{(e)}(t) = A \Theta(t - t_b)\left[1 - \cos\left(\frac{2\pi(t - t_b)}{T_p}\right)\right], \tag{59}$$

with $A = 0.1$, $t_b = 50$ and $T_p = 100$ is applied, input being plotted at the bottom of Fig. 12(a). The solid curve of Fig. 12(c) expresses the $S - (d\mu_1/dt)$ plot, where initial data of S and $d\mu_1/dt$ at $t < 350$ are neglected. In contrast, the dashed curve in Fig. 12(c) shows the $S - (d\mu_1/dt)$ plot in the case of additive noise ($\alpha = 0.0$, $\beta = 0.01$). It is noted that the behavior of the $S - (d\mu_1/dt)$ plot of multiplicative noise is similar to that of additive noises for $d\mu_1/dt < 0$ but rather different for $d\mu_1/dt >$. This is consistent with the result shown in Fig. 11(c) for spike train. It is interesting that the synchronization S seems to correlate with $| d\mu_1/dt |$: S becomes larger for a larger $| d\mu_1/dt |$.

We have discussed the stochastic bifurcation and the synchronization in FN ensembles subjected to additive and/or multiplicative noises. Our calculations have shown that effects of multiplicative noise are rather different from those of additive noise. This may be understood by a simple analysis as follows. Equation (48) is rewritten as

$$\frac{d\mu_1}{dt} = a_3\mu_1^3 + a_2\mu_1^2 + a_1\mu_1 - c\mu_2 + (3a_3\mu_1 + a_2)\gamma_{1,1} + \frac{\alpha^2}{2}\mu_1 + I. \tag{60}$$

For weak noises, we get

$$\gamma_{1,1} \sim D(\alpha^2\mu_1^2 + \beta^2), \tag{61}$$

where D is the coefficient to be determined from Eqs. (48)-(52) but its explicit form is not necessary for our discussion. Substituting Eq. (61) to Eq. (60), we get

$$\frac{d\mu_1}{dt} = a_3'\mu_1^3 + a_2'\mu_1^2 + a_1'\mu_1 - c\mu_2 + I', \tag{62}$$

with

$$a'_3 = a_3(1 + 3D\alpha^2), \tag{63}$$

$$a'_2 = a_2(1 + D\alpha^2), \tag{64}$$

$$a'_1 = a_1 + \frac{\alpha^2}{2} + 3Da_3\beta^2, \tag{65}$$

$$I' = I + Da_2\beta^2. \tag{66}$$

The dynamics of μ_1 and μ_2 is effectively determined by Eqs. (49) and (62). Results for deterministic FN model are given by setting $\alpha = \beta = 0.0$ in Eqs. (62)-(66). We note in Eqs. (63)-(66) that additive noise modifies constant and linear terms, whereas multiplicative noise changes the linear, quadratic and cubic terms. These differences yield the difference in the stochastic bifurcations for additive and multiplicative noises.

The bifurcation diagram of a single FN model with additive noise is discussed in Refs. [6, 11, 12]. It has been shown that the probability distribution of single FN model obeys Gaussian for weak additive noise while for strong noise, it shows a deviation from Gaussian with the bimodal structure [11, 15]. The transition from the Gaussian to non-Gaussian distribution is considered to show the stochastic bifurcation. However, a change in the form of probability density is generally gradual when a parameter of the model is changed. By employing the second-order moment method, Tanabe and Pakdaman [11] have obtained the bifurcation diagram showing a critical current I_c of a single FN model with additive noise, which approaches I_c for the deterministic FN model with decreasing the strength of additive noise. Our transition diagram shown in Fig. 4(b) is different from their transition diagram (Fig. 4 of Ref. [11]) in which the OSC state exists even for a strong additive noise. By solving the FPE by an expansion of the Hermite polynomials, Acebrón, Bulsara and Rappel [12] have shown that the stochastic bifurcation cannot be obtained for a single FN model against our result showing the bifurcation for $N = 1$ [Fig. 4(b)]. Recently, stochastic bifurcations in globally coupled ($N = \infty$) ensembles subjected to additive noise have been discussed in Refs. [12, 15, 17]. It has been shown [12] that contrary to a single element, the bifurcation occurs in globally coupled ensembles as the noise strength is increased. With the use of the moment method, Zaks, Sailer, Schimansky-Geier and Neiman [17] have discussed the bifurcation of coupled ($N = \infty$) FN model in which slow variables (y_i) are subjected to additive noise whereas in our study, fast variables (x_i) are subjected to additive and/or multiplicative noises [Eqs. (1) and (2)]. With increasing the noise intensity, mean field shows a transition from a steady

equilibrium to global oscillations, and then back to another equilibrium for sufficiently strong noise [17].

In the conventional moment (or cumulant) approach, the Gaussian distribution is assumed for calculations of first and second moments, and the moment method is considered to lose its validity for the non-Gaussian distribution [5]-[11],[17]. Our reformulation of the AMM with FPE has revealed that the moment method is free from the Gaussian approximation and that it is valid also for non-Gaussian distribution. Indeed, we have shown in Refs. [34, 35] that the AMM can be well applied to the Langevin model with multiplicative Gaussian noise although its probability distribution generally follows the non-Gaussian [30, 31]. The moment approach is expected to have the wider applicability than having been considered so far.

The transition diagrams shown in Figs. 4(a), 4(c) and 4(e) for multiplicative noise have the asymmetric, peculiar structure, which arises from the assumed form of $G(x) = x$ in Eq. (35). If we alternatively assume $G(x) = 1$, 'multiplicative noise' reduces to additive noise, and its transition diagrams are given by Figs. 4(b), 4(d) and 4(f). Thus the structure of the bifurcation transition diagram depends on the adopted form of $G(x)$. Some preliminary calculations have been made with the use of a form of $G(x) = x \mid x \mid^{s-1}$ by changing the index s: the relevant AMM equations are presented in Appendix A.

Figure 13(a) shows time courses of $S(t)$ for various s with $\alpha = 0.01$, $\beta = 0.0$ and $N = 100$. At a glance, the overall behavior of $S(t)$ seems almost independent of s. We note, however, that $\gamma_{1,1}$ and $\rho_{1,1}$, which are plotted in Figs. 13(b) and 13(c), respectively, much depend on the s value. It has been shown that in the Langevin model, the stationary distribution shows much variety depending on the index s in $G(x) = x \mid x \mid^{s-1}$ [34, 35]. This is expected to be true also for the distribution $p(x, y)$ in the FN model with multiplicative noises. It would be interesting to investigate the stochastic bifurcation by changing the index s, whose study is under consideration.

5   CONCLUSION

We have studied effects of additive and multiplicative noises in single elements and globally-coupled ensembles described by the FN model, by employing AMM reformulated with the use of FPE [34, 13]. The stochastic bifurcation has been examined by a linear analysis of the deterministic AMM equations. The properties of the multiplicative noise in FN neuron ensembles is summarized as follows.

(a) the effect of multiplicative noise on the stochastic bifurcation is different from that of additive noise, and

(b) the effect of multiplicative noise on the synchronization is more significant than that of additive noise.

The item (a) is consistent with the results obtained for other nonlinear systems such as Duffing-Van der Pol model [37, 41]. The effect of noise depends on the type of noises, and it is also model dependent [42]. The item (b) is similar to the item (1) of an ordered state created by multiplicative noises [21]-[29] mentioned in the introduction.

A disadvantage of our AMM is that its applicability is limited to weak-noise cases. In physics, there are many approximate methods which are valid in some limits but which provide us with clear physical picture beyond these limits. One of such examples is the random-phase approximation (RPA) which has been widely employed in solid-state physics. The RPA is valid in the limit of weak interactions. The RPA is, however, used for larger interactions leading to the divergence in the response function, which signifies the occurrence of excitations such as spin waves. We expect that the AMM may be such an approximate method, yielding meaningful qualitative result even for strong noises.

On the contrary, an advantage of the AMM is that we can easily discuss dynamical properties of the finite N-unit stochastic systems. For N-unit FN neuronal ensembles, the AMM yields the eight-dimensional ordinary DEs, while DS and FPE require the $2N$-dimensional stochastic DEs and the $(2N + 1)$-dimensional partial DEs, respectively. Furthermore the calculation of the AMM is much faster than DSs: for example, it is about 2000 times faster than DS calculations with 100 trials for 100-unit FN ensembles. We hope that the AMM may be applied to a wide class of coupled stochastic systems subjected to additive and/or multiplicative noises.

Acknowledgements

Appendix A: AMM equations for $G(x) = x \mid x \mid^{s-1}$

In the case of $G(x)$ given by

$$G(x) \;\; = \;\; x \mid x \mid^{s-1} \qquad (s \geq 0), \qquad\qquad \text{(A1)}$$

AMM equations (51)-(58) become

$$\frac{d\mu_1}{dt} = f_o + f_2\gamma_{1,1} - c\mu_2 + \frac{1}{2}\alpha^2 s\mu_1 \mid \mu_1 \mid^{2s-2} [1 + (s-1)(2s-1) \mid \mu_1 \mid^{-2} \gamma_{1,1}]$$
$$+ \ I^{(e)}, \tag{A2}$$

$$\frac{d\mu_2}{dt} = b\mu_1 - d\mu_2 + e, \tag{A3}$$

$$\frac{d\gamma_{1,1}}{dt} = 2(a\gamma_{1,1} - c\gamma_{1,2}) + \frac{2JN}{Z}(\rho_{1,1} - \gamma_{1,1}) + 2\alpha^2 s(2s-1) \mid \mu_1 \mid^{2s-2} \gamma_{1,1}$$
$$+ \ \alpha^2 \mid \mu_1 \mid^{2s} + \beta^2, \tag{A4}$$

$$\frac{d\gamma_{2,2}}{dt} = 2(b\gamma_{1,2} - d\gamma_{2,2}), \tag{A5}$$

$$\frac{d\gamma_{1,2}}{dt} = b\gamma_{1,1} + (a-d)\gamma_{1,2} - c\gamma_{2,2} + \frac{JN}{Z}(\rho_{1,2} - \gamma_{1,2})$$
$$+ \ \frac{1}{2}\alpha^2 s(2s-1) \mid \mu_1 \mid^{2s-2} \gamma_{1,2}, \tag{A6}$$

$$\frac{d\rho_{1,1}}{dt} = 2(a\rho_{1,1} - c\rho_{1,2}) + 2\alpha^2 s(2s-1)\mu_1^{2s-2}\rho_{1,1} + \frac{\alpha^2 \mid \mu_1 \mid^{2s}}{N} + \frac{\beta^2}{N}, \tag{A7}$$

$$\frac{d\rho_{2,2}}{dt} = 2(b\rho_{1,2} - d\rho_{2,2}), \tag{A8}$$

$$\frac{d\rho_{1,2}}{dt} = b\rho_{1,1} + (a-d)\rho_{1,2} - c\rho_{2,2} + \frac{1}{2}\alpha^2 s(2s-1) \mid \mu_1 \mid^{2s-2} \rho_{1,2}. \tag{A9}$$

Some numerical examples of $\gamma_{11}(t)$ *et al.* for various values of the index s are shown in Fig. 13.

Appendix B: Jacobian matrix of AMM equations

For an analysis of the stochastic bifurcation of finite N-unit FN ensembles, we need the 8×8 Jacobian matrix C expressed with a basis of $(\mu_1, \mu_2, \gamma_{1,1}, \gamma_{2,2}, \gamma_{1,2}, \rho_{1,1}, \rho_{2,2}, \rho_{1,2})$, by

$$C = \begin{bmatrix} f_0' + f_2'\gamma_{1,1} + \alpha^2/2 & -c & f_2 & 0 \\ b & -d & 0 & 0 \\ 2[(f_1' + 3f_3'\gamma_{1,1})\gamma_{1,1} + \alpha^2\mu_1] & 0 & 2(f_1 + 6f_3\gamma_{1,1} + \alpha^2) - 2JN/Z & 0 \\ 0 & 0 & 0 & -2d \\ (f_1' + 3f_3'\gamma_{1,1})\gamma_{1,2} & 0 & b + 3f_3\gamma_{1,2} & -c \\ 2(f_1' + 3f_3'\gamma_{1,1})\rho_{1,1} + 2\alpha^2\mu_1/N & 0 & 6f_3\rho_{1,1} & 0 \\ 0 & 0 & 0 & 0 \\ (f_1' + 3f_3'\gamma_{1,1})\rho_{1,2} & 0 & 3f_3\rho_{1,2} & 0 \end{bmatrix}$$

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$$
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-2c & 2JN/Z & 0 & 0 \\
2b & 0 & 0 & 0 \\
a - d + \alpha^2/2 - JN/Z & 0 & 0 & JN/Z \\
0 & 2(a + \alpha^2) & 0 & -2c \\
0 & 0 & -2d & 2b \\
0 & b & -c & a - d + \alpha^2/2
\end{bmatrix} ,
\tag{B1}
$$

where $a = f_1 + 3f_3\gamma_{1,1}$ and $f_k' = df_k/d\mu_1$, the (1-4)-column and (5-8)-column elements being separately expressed for a convenience of printing.

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